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Relative Invariants of the Polynomial Rings over the Type D_r Quivers

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In this paper we consider the following problem. Let F be a representation of a D_r-type quiver with r vertices and arbitrary directed arrows. For example,

$$\begin{tikzcd} \overset{V_1}{\bullet} \arrow[r, "f_1"] & \overset{V_2}{\bullet} \arrow[r, "f_2"] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \cdots \arrow[r, "f_{r-3}"] & \overset{V_{r-2}}{\bullet} \arrow[r, "f_{r-1}"] & \overset{V_{r-1}}{\bullet}, \\ & & & & & & V_r \arrow[u, "f_r"'] & \end{tikzcd} \tag{F}$$

where V_i is a finite-dimensional vector space over some field k and, if $\overset{V_i}{\bullet} \xrightarrow{f} \overset{V_j}{\bullet}$, f is a linear endomorphism from V_i to V_j. Then we put $V = \bigoplus_{i \to j \text{ in } F} \operatorname{Hom}(V_i, V_j)$ and $G = GL(V_1) \times GL(V_2) \times \cdots \times GL(V_r)$. Then G acts on V naturally, i.e., for $g = (g_1, g_2, ..., g_r) \in G$, the action of G on V is given by $g \cdot f = g_j f g_i^{-1}$, if $\overset{V_i}{\bullet} \xrightarrow{f} \overset{V_j}{\bullet}$.

Let $S(V)$ be the polynomial ring over V. Then the action of G on V naturally extends to the action on $S(V)$. In this paper we consider the problem,

Problem. What are the relative invariants in $S(V)$ with respect to this action?,

and we give an answer to the problem for the D_r-type quivers when the ground field k is the complex number field $\mathbb{C}$.

Namely, we give explicit genrators $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s of the relative invariants in $S(V)$, and show that these generators are algebraically independent.

Our method is based on the combinatorics on Young diagrams, namely explicit calculations of the Littlewood–Richardson coefficients (see Lemma 2, 3, 4) and the standard monomial theory. (See [D-R-S] and the proof of the theorem.)

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This paper is a continuation of [K1], in which we raised the following problem:

Problem 1″ *and* 2″. What are the relative invariants $S(V)^{\mathrm{rel}}$ and the absolute invariants $S(V)^G$ for an arbitrary quiver F? In particular, give a set of explicit generators of $S(V)^{\mathrm{rel}}$ for the finite-type quivers F (namely, F is a quiver whose shape is a Dynkin diagram of A- or D- or E-type) and give a set of explicit generators of $S(V)^{\mathrm{rel}}$ and $S(V)^G$ for the tame-type quivers $\tilde{F}$ (namely, $\tilde{F}$ is a quiver whose shape is a Dynkin diagram of $\tilde{A}$- or $\tilde{D}$- or $\tilde{E}$-type).

In the previous paper [K1] we gave an answer to the above problem for the A_r- and $\tilde{A}_r$-type quivers. (For the exact definition and meanings of finite and tame-type quivers, see [Ka1, Ka3, Ka4, Ga1, Ga2, and B-G-P].)

From the above and the result in [K1], the next problem comes up naturally and seems to be interesting.

Problem. For what quivers do the relative invariants $S(V)^{\mathrm{rel}}$ have algebraically independent generators? More specifically, does this condition (having the algebraically independent generators) characterize the finite and the tame type quivers?

For the A_r-, D_r-, and $\tilde{A}_r$-type quivers, this condition is satisfied.

1. D_n Quivers

In this section we give an answer to the problem for the D_r-type quivers. First let us recall some lemmas in [K1]. Let $V_\lambda^{GL(n)}$ be an irreducible representation space corresponding to a partition λ. (We attach superscript $GL(n)$ to V_λ since we want to indicate which group acts on this space.) Then the irreducible decomposition of $S(M(n, m, \mathbb{C}))$ is given as follows.

LEMMA 1. *As a* $GL(n) \times GL(m)$ *module,*

$$S(M(n, m, \mathbb{C})) = \sum_{\substack{\lambda \in \wp \\ l(\lambda) \leq n, m}} V_\lambda^{GL(n)} \otimes (V_\lambda^{GL(m)})^*,$$

where * *denotes the dual representation.*

LEMMA 2. *For partitions* $\lambda, \mu \in \wp$ *of length at most n, the linear character* $(\det)^i$ *of* $GL(n)$ *occurs in* $V_\lambda^{GL(n)} \otimes (V_\mu^{GL(n)})^*$ *if and only if* $\lambda = \mu + (i^n)$. *Moreover, the multiplicity of* $(\det)^i$ *is at most* 1.

Here $\lambda = \mu + (i^n)$ means that $\lambda_k = \mu_k + i$ for any $k = 1, 2, \ldots,$ and we consider the partitions as infinite sequences by attaching infinite 0's to the ends of the partitions.

LEMMA 3. *For partitions λ, μ of length at most n, the linear character* $(\det)^i$ *of $GL(n)$ occurs in $V_\lambda^{GL(n)} \otimes V_\mu^{GL(n)}$ if and only if $\lambda^{\dagger(i,n)} = \mu$. Moreover, the multiply of* $(\det)^i$ *is at most* 1.

As for the definition of $\lambda^{+(i,n)}$, see the next part of Lemma 4.

We recall some notations. A vertex i in a quiver F is called a "source" if all the arrows connected to i are started from i, and a vertex j is called a "sink" if all the arrows connected to j are terminated at j.

We fix a base of each vector space V_i. Let us denote the dimension of V_i by n_i $(i = 1, 2, \ldots, r)$. Then $S(V)$ can be considered as the polynomial ring in the indeterminates $\{x_{i,j}^{(s)}\}$, where $1 \leqq i \leqq n_{s+1}$, $1 \leqq j \leqq n_s$, and $s = 1, 2, \ldots, r-1$. To be precise, if we substitute some values to $x_{i,j}^{(s)}$'s then the matrix $(x_{i,j}^{(s)})_{i,j}$ corresponds to the homomorphism f_s with respect to the above basis.

If $\overset{V_s}{\bullet} \xrightarrow{f_s} \overset{V_{s+1}}{\bullet}$, let $M_{s+1,s}$ be the matrix $(x_{i,j}^{(s)})_{i,j}$. ($n_{s+1} \times n_s$ matrix whose (i,j)-th coefficient is the indeterminante $x_{i,j}^{(s)}$.) If $\overset{V_s}{\bullet} \xleftarrow{f_s} \overset{V_{s+1}}{\bullet}$, let $M_{s,s+1}$ be the matrix $(x_{i,j}^{(s)})_{i,j}$. ($n_s \times n_{s+1}$ matrix whose (i,j)-th coefficient is the indeterminate $x_{i,j}^{(s)}$.)

First let us recall some notations of the A_r-type quivers used in [K1]. Let

u, v $(u < v)$ be vertices in F such that there are no sinks and sources between them. Then we define the matrix by $M_{v,u} = M_{v,v-1} M_{v-1,v-2} \cdots M_{u+1,u}$, if $\overset{u}{\bullet} \to \bullet \to \bullet \to \cdots \to \overset{v}{\bullet}$ and $M_{u,v} = M_{u,u+1} M_{u+1,u+1} \cdots M_{v-1,v}$, if $\overset{u}{\bullet} \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \overset{v}{\bullet}$.

DEFINITION 1. For any p, q with $1 \leqq p \leqq q \leqq r$ and $n_p = n_q$, if there is no sink and source between p and q, we define the polynomial $p_{q,p}$ by

$$P_{q,p} = \begin{cases} \det(M_{q,p}) & \text{if the arrows directed from } p \text{ to } q \\ \det(M_{p,q}) & \text{if the arrows directed from } q \text{ to } p \end{cases}$$

and call these polynomials by determinantal invariants.

$P_{q,p}$ is a relative invariant and $P_{q,p} \neq 0$ if and only if for any v $(p < v < q)$, $n_v \geqq n_p = n_q$. *Moreover, if a pair* (p, q) satisfies the condition that $n_v > n_p = n_q$ for any v $(p < v < q)$, then we call the determinantal invariant $P_{q,p}$ *primitive.* Clearly, any determinantal invariant can be written as a product of the primitive ones.

If there exist sinks and sources between the vertices p and q, then the determinantal invariant is defined as follows (see [K1, §4] for the details).

For example, let $u_1, u_2, u_3, \ldots, u_k$ $(p < u_1 < u_2 < \cdots < u_k < q)$ be the sourves between p and let $v_1, v_2, v_3, \ldots, v_l$ $(p < v_1 < v_2 < \cdots < v_l < q)$ be the sinks between p and q (l can be $k+1$ or k or $k-1$) and assume that they are located as follows.

$$p < u_1 < v_1 < u_2 < \cdots < u_k < v_k < q$$

$$\overset{p}{\bullet} \leftarrow \bullet \leftarrow \overset{u_1}{\bullet} \rightarrow \bullet \rightarrow \overset{v_1}{\bullet} \leftarrow \overset{u_2}{\bullet} \rightarrow \cdots \rightarrow \overset{v_k}{\bullet} \leftarrow \bullet \leftarrow \overset{q}{\bullet}.$$

In this case, we define the matrix M by

$$M = \begin{pmatrix} M_{p,u_1} & 0 & 0 & 0 & \cdots & 0 \\ M_{v_1,u_1} & M_{v_1,u2} & 0 & 0 & \cdots & 0 \\ 0 & M_{v_2,u_2} & M_{v_2,u_3} & 0 & \cdots & 0 \\ 0 & 0 & M_{v_3,u_3} & \ddots & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & M_{v_k,u_k} & M_{v_k,q} \end{pmatrix}.$$

Then M is an $(n_p + n_{v_1} + n_{v_2} + \cdots + n_{v_k}) \times (n_{u_1} + n_{u_2} + \cdots + n_{u_k} + n_q)$ matrix. If $n_p + n_{v_1} + n_{v_2} + \cdots + n_{v_k} = n_{u_1} + n_{u_2} + \cdots + n_{u_k} + n_q$, we can take the determinant of M.

If $\det(M) \neq 0$, then $P_{q,p} = \det(M)$ is a relative invariant of weight

$$(0, 0, ..., \underset{\hat{p}}{1}, 0, ..., \underset{\widehat{u_1}}{-1}, ..., \underset{\widehat{v_1}}{1}, ..., 0, ..., \underset{\widehat{v_k}}{1}, 0, ..., 0, \underset{\widehat{q}}{-1}, 0, ..., 0).$$

The necessary and sufficient conditions for $\det(M) \neq 0$ is given by

$$\begin{aligned} n_p &\leqq n_{p+1}, n_{p+2}, ..., n_{u_1}, \\ n_{u_1} - n_p &\leqq n_{u_1+1}, n_{u_1+2}, ..., n_{v_1}, \\ n_{v_1} - n_{u_1} + n_p &\leqq n_{v_1+1}, n_{v_1+2}, ..., n_{u_2}, \\ n_{u_2} - n_{v_1} + n_{u_1} - n_p &\leqq n_{u_2+1}, n_{u_2+2}, ..., n_{v_2}, \\ \vdots \qquad &\leqq \qquad \vdots \\ n_{v_k} - n_{u_k} + n_{v_{k-1}} - \cdots + n_p &\leqq n_{v_k+1}, n_{v_k+2}, ..., n_{q-1}. \end{aligned}$$

A determinantal invariant $P_{q,p} = \det(M)$ is called "*primitive*" if the inequalities in the above hold strictly, namely if $n_p, n_{p+1}, ..., n_q$ satisfies

$$\begin{aligned} n_p &< n_{p+1}, n_{p+2}, ..., n_{u_1}, \\ n_{u_1} - n_p &< n_{u_1+1}, n_{u_1+2}, ..., n_{v_1}, \\ n_{v_1} - n_{u_1} + n_p &< n_{v_1+1}, n_{v_1+2}, ..., n_{u_2}, \\ n_{u_2} - n_{v_1} + n_{u_1} - n_p &< n_{u_2+1}, n_{u_2+2}, ..., n_{v_2}, \\ \vdots \qquad &< \qquad \vdots \\ n_{v_k} - n_{u_k} + n_{v_{k-1}} - \cdots + n_p &< n_{v_k+1}, n_{v_k+2}, ..., n_{q-1}. \end{aligned}$$

Any determinantal invariant can be decomposed into a product of the primitive ones.

For the cases in which the sources and sinks between p and q are located differently, the matrix whose determinant gives a determinantal invariant is obtained by arranging the matrices $M_{v,u}$ and $M_{v',u}$ vertically at the source u (v and v' are the sinks adjacent to u) and by arranging the matrices $M_{v,u}$ and $M_{v,u'}$ horizontally at the sink v (u and u' are the sources adjacent to v) and by putting 0 matrices at the other places. The primitiveness of them is defined by inequalities similar to the above. (See [K, §4] for the details.)

In any cases the relative invariants for the A_r-type quivers are the monomials of the primitive determinantal invariants and the primitive ones are algebraically invariants.

We come back to the situation of Problem 1. First we treat the simplest case. Let F be a one-way directed quiver of type D_n.

$$\begin{tikzcd}
\overset{V_1}{\bullet} \arrow[r, "f_1"] & \overset{V_2}{\bullet} \arrow[r, "f_2"] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r, "f_{r-3}"] & \overset{V_{r-2}}{\bullet} \arrow[r, "f_{r-1}"] \arrow[d, "f_r"] & \overset{V_{r-1}}{\bullet} . \\
 & & & & & & V_r &
\end{tikzcd} \tag{F}$$

Then $S(V)$ is decomposed as follows.

$$\begin{aligned} S(V) &= S\left(\bigoplus_{i=1}^{r-2} \operatorname{Hom}(V_i, V_{i+1}) \oplus \operatorname{Hom}(V_{r-2}, V_r)\right) \\ &= \bigotimes_{i=1}^{r-2} S(\operatorname{Hom}(V_i, V_{i+1})) \otimes S(\operatorname{Hom}(V_{r-2}, V_r)). \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} = \sum_{\substack{\lambda^{(1)}, \lambda^{(2)}, \dots, \widehat{\lambda_{(r-2)}}, \lambda_{(r-1)}, \lambda_{(r)} \\ l(\lambda^{(1)}) \leqq n_1, n_2 \\ l(\lambda^{(2)}) \leqq n_2, n_3, \\ \vdots \\ l(\lambda^{(r-1)}) \leqq n_{r-1}, n_r,}} & (V_{\lambda^{(1)}}^{GL(n_1)})^* \otimes V_{\lambda^{(1)}}^{GL(n_2)} \\ & \otimes (V_{\lambda^{(2)}}^{GL(n_2)})^* \otimes V_{\lambda^{(2)}}^{GL(n_3)} \otimes \cdots \otimes (V_{\lambda^{(r-3)}}^{GL(n_{r-3})})^* \\ & \otimes V_{\lambda^{(r-3)}}^{GL(n_{r-2})} \otimes (V_{\lambda^{(r-1)}}^{GL(n_{r-2})})^* \otimes V_{\lambda^{(r-1)}}^{GL(n_{r-1})} \\ & \otimes (V_{\lambda^{(r)}}^{GL(n_{r-2})})^* \otimes V_{\lambda^{(r)}}^{GL(n_r)}. \end{aligned} \tag{S}$$

Then

$$(V_{\lambda^{(r-1)}}^{GL(n_{r-2})})^* \otimes (V_{\lambda^{(r)}}^{GL(n_{r-2})})^* = \sum_{\xi} LR_{\lambda^{(r-1)}, \lambda^{(r)}}^{\xi} (V_{\xi}^{GL(n_{r-2})})^*.$$

First we prove the following lemma.

LEMMA 4. *The Littlewood–Richardson coefficient* $LR^{\xi}_{(i_{r-1}^{n_{r-1}})(i_r^{n_r})}$ *is* 0 *or* 1.

Proof. We can assume that $\xi \supseteq (i_{r-1}^{n_{r-1}})$ and $\xi \supseteq (i_r^{n_r})$, since if not, it is known that $LR^{\xi}_{(i_{r-1}^{n_{r-1}})(i_{r-1}^{n_{r-1}})} = 0$. (See [M, p. 40].) The number of the connected components of the skew diagram $\xi/(i_{r-1}^{n_{r-1}})$ is 0 or 1 or 2. Let us define the partition $\eta = (\eta_1, \eta_2, ..., \eta_{n_{r-1}})$ by $\eta_1 = \xi_1 - i_{r-1}$, $\eta_2 = \xi_2 - i_{r-1}$, $\eta_3 = \xi_3 - i_{r-1}, ..., \eta_{n_{r-1}} = \xi_{n_{r-1}} - i_{r-1}$, and the partition $\mu = (\mu_1, \mu_2, ..., \mu_{l-n_{r-1}})$ by $\mu_1 = \xi_{n_{r-1}+1}$, $\mu_2 = \xi_{n_{r-1}+2}, ..., \mu_{l-n_{r-1}} = \xi_l$, where $\xi = (\xi_1, \xi_2, ..., \xi_l)$.

We insert i_r symbol 1's, and i_r symbol 2's, ..., and i_r symbol n_r's in the diagrams η and μ satisfying the Littlewood–Richardson rules. It is easy to see that there is the unique way to put the symbols in the η part of the skew diagram $\xi/(i_{r-1}^{n_{r-1}})$. Namely we must insert symbol 1's into the first row of η and symbol 2's into the second row of η, ... and symbol n_{r-1}'s into the last row of η. Therefore we have only to prove that there is at most one way to put $i_r - \eta_1$ symbol 1's and $i_r - \eta_2$ symbol 2's, ..., $i_r - \eta_{n_r}$ symbol n_r's in the μ part of the skew diagram $\xi/(i_{r-1}^{n_{r-1}})$ satisfying the Littlewood–Richardson rules.

Assume that we have a way. Let us denote the occurrences of symbol i in the sth row of μ by k_i^s. Then the symbol 1's must occur in the first row of μ, hence $k_1^1 = i_r - \eta_1$. Next we consider the ways to put symbol 2's in the μ. The lattice permutation says that $k_2^1 + \eta_2 \leqq \eta_1$. Also in the second row of μ, the symbol 2's must occur only in the squares lying just under the squares in which the symbol 1's are already inserted. Therefore we have $k_2^2 \leqq k_1^1 = i_r - \eta_1$. Hence we have

$$k_2^1 + k_2^2 \leqq i_r - \eta_2.$$

On the other hand, the symbol 2 must not appear under the second row, so $k_2^1 + k_2^2 = i_r - \eta_2$. Hence we have equalities

$$k_2^1 = \eta_1 - \eta_2, \qquad k_2^2 = i_r - \eta_1.$$

We prove the next claim by induction on u.

Claim.

$$k_u^t = \eta_{u-t} - \eta_{u-t+1},$$

where $t = 1, 2, ..., u$ and we consider $\eta_0 = i_r$.

Proof. From the above, the claim holds for $u = 1$ and $u = 2$. Assume that the claim holds for any $u \leqq v$, where v is an integer greater than 1. We prove the claim for $u=v+1$. From the induction hypothesis, in the Young diagram μ symbols $1, 2, \ldots, v$ should be placed as follows:

$$\begin{array}{cccccccccccccc}
1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & 3 & \cdots & 3 & \cdots & \cdots & v \quad v \cdots v \\
2 & 2 & \cdots & 2 & 3 & 3 & \cdots & 3 & 4 & \cdots & 4 & \cdots & \cdots & \\
3 & 3 & \cdots & 3 & 4 & 4 & \cdots & 4 & \vdots & \cdots & \vdots & & & \\
4 & 4 & \cdots & 4 & \vdots & \vdots & \cdots & \vdots & v & \cdots & v & & & \\
\vdots & \vdots & \cdots & \vdots & v & v & \cdots & v & & & & & & \\
v & v & \cdots & v & & & & & & & & & &
\end{array}$$

The lattice permutation says that

$$k^1_{v+1} \leqq \eta_v - \eta_{v+1}.$$

Also, $v+1$ can be placed beneath the symbol v's only since $k^i_u = k^{i-1}_{u-1}$. Therefore for $i=2, 3, \ldots, v+1$ we have

$$k^i_{v+1} \leqq k^{i-1}_v = \eta_{v-i+1} - \eta_{v-i+2}.$$

So we have the following equality:

$$k^1_{v+1} + k^2_{v+1} + \cdots + k^{v+1}_{v+1} \leqq i_r - \eta_{v+1}.$$

Since $k^1_{v+1} + k^2_{v+1} + \cdots + k^{v+1}_{v+1} = i_r - \eta_{v+1}$, we have

$$k^i_{v+1} = k^{i-1}_v = \eta_{v-i+1} - \eta_{v-i+2}.$$

The claim is proved.

This claim tells us that if possible we have only one way to put symbols in the Young diagram μ. The lemma is proved.

If $LR^{\xi}_{(i_{r-1}^{n_{r-1}})(i_{r-1}^{n_{r-1}})} = 1$, the above proof shows that ξ is given as follows. From the symmetry of the Dynkin diagram D_n, we can assume that $n_r \leqq n_{r-1}$. Let η and μ be the partitions defined in the above proof. We prepare notation. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be partition of length at most n and let i be an integer with $i \geqq \lambda_1$. We define a partition $\lambda^{\dagger(i,n)}$ by

$$\lambda^{\dagger(i,n)} = (i-\lambda_n, i-\lambda_{n-1}, \ldots, i-\lambda_1).$$

Then μ must be equal to $\eta^{\dagger(i_r, n_r)}$. Namely, $\eta_1 \leqq i_r$, $l(\eta) \leqq n_r$, and $\mu_1 = i_r - \eta_{n_r}$, $\mu_2 = i_r - \eta_{n_r-1}$, $\mu_3 = i_r - \eta_{n_r-2}, \ldots, \mu_s = i_r - \eta_{n_r-s+1}, \ldots, \mu_{n_r} = i_r - \eta_1$.
Also, if $l(\xi) > n_{r-1}$, $\xi/(i_{r-1}^{n_{r-1}})$ has two connected components. If not, the skew diagram $\xi/(i_{r-1}^{n_{r-1}})$ is connected and then we can find a column of ξ which also lies in the skew diagram $\xi/(i_{r-1}^{n_{r-1}})$ and has its length greater than

n_{r-1}. But it is impossible to put symbols $1, 2, \ldots, n_r$ in this column since $n_r \leqq n_{r-1}$.

Conversely, if $l(\eta) \leqq n_r$ $(\leqq n_{r-1})$, $\eta_1 \leqq i_r$, $\mu = \eta^{\dagger(i_r, n_r)}$, we can define the Young diagram ξ by attaching η and μ to the $(i_{r-1}^{n_{r-1}})$. Namely, we put $\xi_1 = i_{r-1} + \eta_1$, $\xi_2 = i_{r-1} + \eta_2, \ldots, \xi_{n_{r-1}} = i_{r-1} + \eta_{n_{r-1}}$, $\xi_{n_{r-1}+1} = \mu_1$, $\xi_{n_{r-1}+2} = \mu_2, \ldots, \xi_l = \mu_{l-n_{r-1}}$. Then we have $LR^{\xi}_{(i_r^{n_r})(i_{r-1}^{n_{r-1}})} = 1$. In this case we say two columns in ξ are complementary to each other if the sum of the lengths of these two columns is equal to $n_{r-1} + n_r$.

If we put $\xi = \lambda^{(r-2)}$, then by Lemma 2 and the above decomposition the relative invariants corresponding to the sequences of Young diagrams $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ are given as follows:

(RS1) $\lambda^{(1)} = (i_1^{n_1})$, $\lambda^{(2)} = \lambda^{(1)} + (i_2^{n_2}), \ldots, \lambda^{(k)} = \lambda^{(k-1)} + (i_k^{n_k}), \ldots, \lambda^{(r-2)} = \xi = \lambda^{(r-3)} + (i_{r-2}^{n_{r-2}})$, $\lambda^{(r-1)} = (i_{r-1}^{n_{r-1}})$, $\lambda^{(r)} = (i_r^{n_r})$, where $i_k \in \mathbb{Z}$.

(RS2) $\lambda^{(k)}$ is a Young diagram for any $k = 1, 2, \ldots, r$.

(RS3) $l(\lambda^{(k)}) \leqq \mathrm{Minimum}(n_k, n_{k+1})$ for any $k = 1, 2, \ldots, r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$

(RS4) $LR^{\xi}_{\lambda^{(r-1)}, \lambda^{(r)}} \neq 0$.

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ satisfying all the above conditions "admissible sequences of Young diagrams (for the relative invariants)."

The decomposition (S) and the lemmas imply that for each admissible sequence we have only one (up to scalars) relative invariants in $S(V)$, and then the weight of the relative invariant is given by $\mathfrak{k} = (-i_1, -i_2, \ldots, -i_{r-2}, i_{r-1}, i_r)$.

Conversely, for any sequence $(a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r$, define

$$\begin{aligned}
\lambda^{(1)} &= ((-a_1)^{n_1}),\\
\lambda^{(2)} &= \lambda^{(1)} + ((-a_2)^{n_2}),\\
&\vdots\\
\lambda^{(r-2)} &= \lambda^{(r-3)} + ((-a_{r-2})^{n_{r-2}}),\\
\lambda^{(r-1)} &= (a_{r-1}^{n_{r-1}}),\\
\lambda^{(r)} &= (a_r^{n_r}),
\end{aligned}$$

and then if $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ satisfies the conditions (RS1), (RS2), (RS3), and (RS4) we call a sequence $(a_1, a_2, \ldots, a_r)$ *an admissible weight.* By $S(V)^{(a_1, a_2, \ldots, a_r)}$ we denote the relative invariants with weight $(a_1, a_2, \ldots, a_r)$ in $S(V)$. Then we summarize the above argument.

Proposition 1. (1) $\dim S(V)^{(a_1, a_2, \ldots, a_r)} \leqq 1$.

(2) *$\dim S(V)^{(a_1, a_2, \ldots, a_r)} = 1$ if and only if $(a_1, a_2, \ldots, a_r)$ is an admissible weight.*

Let us trace a column in an admissible sequence λ which survives until the vertex $r-2$. We put $(\lambda^{(r-2)})' = (\xi)' = (s_1, s_2, ..., s_u)$. Let this column which we are chasing be attached at the vertex p. According to Lemma 4, since $LR^{\xi}_{\lambda^{(r-1)}, \lambda^{(r)}} = 1$, there must be the complementary column of length n_q in ξ attached at the vertex q. From the definition of complementary columns we have $n_p + n_q = n_{r-1} + n_r$. We may assume $p < q$, therefore $n_q \geqq n_p$. Also from the condition (RS3) and the assumption that these two columns survive until the vertex $r-2$, we have $n_p \leqq n_{p+1}, n_{p+2}, ..., n_{r-2}$ and $n_q \leqq n_{q+1}, n_{q+2}, ..., n_{r-2}$.

Let us define

$$\begin{aligned}\phi_{p,q,r-1,r} = \sum \operatorname{sgn}(I, \bar{I}) \operatorname{sgn}(J, \bar{J}) &(M_{r,r-2} M_{r-2,p})_{I,[n_p]} \\ &\times (M_{r,r-2} M_{r-2,q})_{\bar{I},J} (M_{r-1,r-2} M_{r-2,q})_{[n_{r-1}],\bar{J}},\end{aligned}$$

where $|I| = n_p$ and I is a subset of $[n_r] = \{1, 2, ..., n_r\}$ and J is a subset of $[n_p] = \{1, 2, ..., n_p\}$. The above sum runs over the entire set I, J such that $I \cup \bar{I} = [n_r]$ (disjoint union), and $J \cup \bar{J} = [n_q]$ (disjoint union). $\operatorname{sgn}(I, \bar{I})$ denotes the signature of the perumation of

$$w = \begin{pmatrix} 1 & 2 & \cdots & n_p & n_p+1 & \cdots & n_r - 1 & n_r \\ i_1 & i_2 & \cdots & i_{n_p} & i_{n_p+1} & \cdots & i_{n_r-1} & i_{n_r} \end{pmatrix},$$

where $I = \{i_1, i_2, ..., i_{n_p}\}$, $(i_1 < i_2 < \cdots < i_{n_p})$, and $\bar{I} = \{i_{n_p+1}, ..., i_{n_r-1}, i_{n_r}\}$, $(i_{u+1} < \cdots < i_{n_r-1} < i_{n_r})$. The $\operatorname{sgn}(J, \bar{J})$'s have the same meaning. $(M_{r,r-2} M_{r-2,p})_{I,[n_p]}$ (where $n_p = |I|$) denotes the n_p-minor of $M_{r,r-2} M_{r-2,p}$ whose rows are the i_u-th rows of $M_{r,r-2} M_{r-2,p}$ ($u = 1, 2, ..., n_p$ and $I = \{i_1, i_2, ..., i_{n_p}\}$) and whose columns are all the columns of $M_{r,r-2} M_{r-2,q}$. $(M_{r,r-2} M_{r-2,q})_{\bar{I},J}$ denotes the $|\bar{I}| = |J|$ minor of $M_{r,r-2} M_{r-2,p}$ whose row indices are in $\bar{I}$ and whose column indices are in J. $(M_{r-1,r-2} M_{r-2,q})_{[n_{r-1}],\bar{J}}$ is defined similarly.

This $\phi_{q,p,r-1,r}$ also has a determinantal expression given as follows.

$$\begin{aligned}\phi_{q,p,r-1,r} &= \det \begin{pmatrix} M_{r,r-2} M_{r-2,p} & M_{r,r-2} M_{r-2,q} \\ 0 & M_{r-1,r-2} M_{r-2,q} \end{pmatrix} \\ &= (-1)^{n_r n_{r-1} - n_p^2} \det \begin{pmatrix} M_{r-1,r-2} M_{r-2,p} & M_{r-1,r-2} M_{r-2,q} \\ 0 & M_{r,r-2} M_{r-2,q} \end{pmatrix},\end{aligned}$$

where $M_{r-2,p}$ is the matrix defined by

$$M_{r-2,p} = M_{r-2,r-3} M_{r-3,r-4} \cdots M_{p+1,p}.$$

By substituting the special values into $x^{(s)}_{i,j}$, we can see easily that $\phi_{q,p,r-1,r} \neq 0$. (See [K1].) It is clear that this $\phi_{q,p,r-1,r}$ is a relative invariant. If $n_p < n_{p+1}, n_{p+2}, \ldots, n_{r-2}$ and $n_q < n_{q+1}, n_{q+2}, \ldots, n_{r-2}$, we call this $\phi_{q,p,r-1,r}$ *primitive*. Any $\phi_{q,p,r-1,r}$ can be written as the product of the primitive $\phi_{i,j,r-1,r}$'s and the primitive $P_{i,j}$'s.

Then our theorem is as follows.

THEOREM. *The relative invariants in $S(V)$ amount to being the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s, and the primitive relative invariants are algebraically independent.*

We prove this theorem later for the D_r-type quivers with arbitrary directed arrows, so we omit the proof here.

We move to the general cases. Let F be a quiver in which the arrows at the branching vertex $r-2$ are directed as follows and the other arrows are directed arbitrarily.

Case: Ordinary at $r-2$ (2 arrows started from $r-2$ to r and $r-1$).

$$\begin{tikzcd}[column sep=small] \overset{p}{\bullet} \arrow[r] & \bullet \arrow[r] & \bullet & \cdots \arrow[l] \arrow[r] & \overset{q}{\bullet} \arrow[r] & \bullet & \cdots \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \overset{r-2}{\bullet} \arrow[r] \arrow[d] & \overset{r-1}{\bullet} \\ &&&&&&&&& r & \end{tikzcd}$$

We trace a column in an admissible sequence. Let this column attach at the vertex p and survive until the vertex $r-2$. Then from Lemma 4 there is a complementary column which is attached at the vertex q.

As in the A_r-type quivers, according to the distribution of the sources and the sinks between the vertices p and q, we must divide the cases. But as in the cases of the A_r-type quivers, a matrix whose determinant gives a primitive invariant is obtained by arranging the matrices $M_{v,u}$ and $M_{v',u}$ vertically at the source u (v and v' are the sinks adjacent to u) and by arranging the matrices $M_{v,u}$ and $M_{v,u'}$ horizontally at the sink v (u and u' are the sources adjacent to v) and by putting 0 matrices at the other places.

So for the D_r-type quivers we only give a primitive invariant for an exemplified case, since for the other cases primitive invariants are defined in just the same way.

For example, in the above quiver let the sources and the sinks between p and $r-2$ be located as follows:

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < v_s < u_s < r-2.$$

If $n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p + n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q = n_{r-1} + n_r$, then we define the matrix M in the following way.

In the case of $n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q>n_r$ and $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p<n_{r-1}$, let

$$M=\begin{pmatrix} M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{v_s,u_{s-1}} & M_{v_t,u_s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{r,r-2}M_{r-2,u_s} & M_{r,r-2}M_{r-2,u_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{r-1,r-2}M_{r-2,u_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{v_t,u_t} & M_{v_t,q} \end{pmatrix}.$$

If $n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q=n_r$, hence $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p=n_{r-1}$, the situation reduces to the A_r cases.

This $\phi_{q,p,r-1,r}=\det(M)$ is called primitive if

$$\begin{aligned} n_p &< n_{p+1}, n_{p+2}, \ldots, n_{v_1}, \\ n_{v_1}-n_p &< n_{v_1+1}, n_{v_1+2}, \ldots, n_{u_1}, \\ n_{u_1}-n_{v_1}+n_p &< n_{u_1+1}, n_{u_1+2}, \ldots, n_{v_2}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s}-n_{v_s}+\cdots+n_p &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2} \end{aligned}$$

and

$$\begin{aligned} n_q &< n_{q+1}, n_{q+2}, \ldots, n_{v_t}, \\ n_{v_t}-n_q &< n_{v_t+1}, n_{v_t+2}, \ldots, n_{u_t}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s}-n_{v_s}+\cdots+n_q &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}. \end{aligned}$$

By substituting the special values to $x_{i,j}^{(s)}$, we can see easily that the primitive $\phi_{q,p,r-1,r}$ is nonzero.

We also define the primitive invariants $\phi_{q,p,r-1,r}$'s for the other cases in which the sinks and sources between p and q and $r-2$ are located in the different ways.

Then we have

THEOREM. *The relative invariants in $S(V)$ amount to being the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s, and the primitive relative invariants are algebraically independent.*

Proof. In this case the sequences of Young diagrams $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ corresponding to the relative invariants $S(V)^{\mathrm{rel}}$ in $S(V)$ are given as follows.

(RS1) $\lambda^{(1)}=(i_1^{n_1})$, and inductively $\lambda^{(s)}$ $(2 \leqq s \leqq r-3)$ are obtained as follows.

If (1) $\overset{s-1}{\bullet} \rightarrow \overset{s}{\bullet} \rightarrow \overset{s+1}{\bullet}$ or (2) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)}=\lambda^{(s-1)}+(i_s^{n_s})$.

If (3) $\overset{s-1}{\bullet} \rightarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$ or (4) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \rightarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)}=\lambda^{\dagger(i_s, n_s)}$, where $i_s \in \mathbb{Z}$.

$\lambda^{(r-2)}=\xi=\lambda^{(r-3)}+(i_{r-2}^{n_{r-2}})$, $\lambda^{(r-1)}=(i_{r-1}^{n_{r-1}})$, $\lambda^{(r)}=(i_r^{n_r})$.

(RS2) $\lambda^{(s)}$ is a Young diagram for any s.

(RS3) $l(\lambda^{(k)}) \leqq \text{minimum}(n_k, n_{k+1})$ for any $k=1, 2, ..., r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$.

(RS4) $LR^{\xi}_{\lambda^{(r-1)}, \lambda^{(r)}} \neq 0$.

The weight of $GL(V_s)$ corresponding to the above sequence is given by i_s for cases (2) and (3) in (RS1) and by $-i_s$ for cases (1) (4) in (RS1).

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(r-1)}$ satisfying all the above conditions "admissible sequences of Young diagrams (for the relative invariants)."

We prove the theorem by induction on $|\mathfrak{k}|=\sum_i |k_i|$. We are chasing a column in an admissible sequence of Young diagrams corresponding to a relative invariant. Namely let this column attach at the vertex p. Then this column will vanish at the vertex q or survive until the last vertices r and $r-1$. In the former case the proof of the type A_r works well. In the latter case if $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p=n_r$ or n_{r-1}, then also the proof of the A_r type works well. If not, at the vertex $r-2$ the condition (RS4) and Lemma 4 tell us that there must be the complementary column in ξ which attaches to this admissible sequence at the vertex q satisfying $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p+n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q=n_{r-1}+n_r$ and $n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q>n_r$ and $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p<n_{r-1}$. Then we define a new sequence $\tilde{\lambda}$'s of Young diagrams obtained by erasing the above two columns from the λ's. Then this sequence is admissible and if we denote the corresponding weight by $\tilde{\mathfrak{k}}$, then $|\tilde{\mathfrak{k}}|=|\mathfrak{k}|-2s-2(s-t+1)-2$. Therefore, from the induction hypothesis, there exists a monomial $\tilde{f}$ in $\phi_{u,v,r-1,r}$'s and $P_{u,v}$'s, whose weight is given by $\tilde{\mathfrak{k}}$. Hence $\phi_{p,q,r-1,r}\tilde{f}$ is a relative invariant with weight $\mathfrak{k}$ corresponding to the weight spaces λ's. From the Proposition 1, we prove the former part of Theorem 1.

We prove that $\phi_{p,q,r-1,r}$'s and $P_{j,i}$'s are algebraically independent.

Assume that there exists some algebraic relations

$$f(P_{i_1,j_1}, P_{i_2,j_2}, ..., P_{i_s,j_s}, \phi_{p_1,q_1,r-1,r}, \phi_{p_2,q_2,r-1,r}, ..., \phi_{p_t,q_t,r-1,r})=0$$

between them, where $P_{i_1,j_1}, ..., P_{i_s,j_s}$, and $\phi_{p_1,q_1,r-1,r}, ..., \phi_{p_t,q_t,r-1,r}$ are primitive. We can choose an $f \neq 0$ such that each monomial occuring in f belongs to the same sequence of Young diagrams $\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(r)}$. If we expand P_{i_w, j_w} and $\phi_{p_k, q_k, r-1, r}$'s into the sum of the products of minors of $M_{i, i-1}$ or $M_{i-1, i}$, the standard monomial theory tells us that for a fixed vertex i, primitive $\phi_{p_k, q_k, r-1, r}$'s ($p \leqq i$) and primitive P_{i_w, j_w}'s ($j_w < i < i_w$) contribute to the $\lambda^{(i)}$ by the columns whose lengths are different from each other. (See [D-R-S, C-P].) Therefore the exponents of primitive $\phi_{p_k, q_k, r-1, r}$'s and primitive P_{i_w, j_w}'s in the monomials in f are determined uniquely, hence f should be a monomial. This is a contradiction since $S(V)$ is a domain. The theorem is proved.

We move to the cases in which the directions of the arrows at the branching vertex $r-2$ are different from the above case. Let F be a quiver defined by

Case: Ordinary at $r-2$ (only 1 arrow started from $r-2$ to $r-1$).

$$
\begin{tikzcd}
\overset{p}{\bullet} \arrow[r] & \bullet \arrow[r] & \overset{v_1}{\bullet} & \overset{u_1}{\bullet} \arrow[l] & \bullet \arrow[l] \arrow[r] & \cdots & \overset{u_s}{\bullet} \arrow[l] \arrow[r] & \overset{r-2}{\bullet} \arrow[r] & \overset{r-1}{\bullet} \\
 & & & & & & & \underset{r.}{} \arrow[u] &
\end{tikzcd}
$$

Then $S(V)$ decomposes as

$$
\begin{aligned}
S(V) = & \sum_{\substack{\lambda^{(1)}, \lambda^{(2)}, \dots, \widehat{\lambda_{(r-2)}}, \lambda_{(r-1)}, \lambda_{(r)} \\ l(\lambda^{(1)}) \leqq n_1, n_2 \\ \vdots \\ l(\lambda^{(r-3)}) \leqq n_{r-3}, n_{r-2}, \\ l(\lambda^{(r-1)}) \leqq n_{r-1}, n_{r-2}, \\ l(\lambda^{(r)}) \leqq n_r, n_{r-2},}} (V_{\lambda^{(1)}}^{GL(n_1)})^* \otimes V_{\lambda^{(1)}}^{GL(n_2)} \\
& \otimes (V_{\lambda^{(2)}}^{GL(n_2)})^* \otimes V_{\lambda^{(2)}}^{GL(n_3)} \otimes \cdots \otimes (V_{\lambda^{(r-3)}}^{GL(n_{r-3})})^* \\
& \otimes V_{\lambda^{(r-3)}}^{GL(n_{r-2})} \otimes (V_{\lambda^{(r-1)}}^{GL(n_{r-2})})^* \otimes V_{\lambda^{(r-1)}}^{GL(n_{r-1})} \\
& \otimes V_{\lambda^{(r)}}^{GL(n_{r-2})} \otimes (V_{\lambda^{(r)}}^{GL(n_r)})^*. \qquad \text{(S)}
\end{aligned}
$$

Hence $l(\lambda^{(r-1)}) \leqq n_{r-2}$ and $\lambda^{(r-1)} = ((i_{r-1})^{n_{r-1}})$. If $n_{r-2} < n_{r-1}$, then i_{r-1} must be 0. In this case the problem reduces to the A_r-type quivers. (See [K1].) So the same is true for n_r and n_{r-2}. We have only to give an answer to the problem in the case of $n_{r-2} \geqq n_{r-1}, n_r$. So from now on we always assume that $n_{r-2} \geqq n_{r-1}, n_r$.

We show that the linear characters $(\det)^k$ of $GL(n_{r-2})$ occur at most once in the space $V_{\lambda^{(r-3)}}^{GL(n_{r-2})} \otimes V_{\lambda^{(r)}}^{GL(n_{r-2})} \otimes V_{\lambda^{(r-1)}}^{GL(n_{r-2})})^*$.

If we put

$$
V_{\lambda^{(r-3)}}^{GL(n_{r-2})} \otimes V_{\lambda^{(r)}}^{GL(n_{r-2})} = \sum_{\xi} LR_{\lambda^{(r-3)}, \lambda^{(r)}}^{\xi} V_{\xi}^{GL(n_{r-3})},
$$

then $\lambda^{(r-1)}$ must be given by $\lambda^{(r-1)} = \xi + (k^{n_{r-2}})$ for some integer k, therefore k is determined uniquely by λ's since $|\lambda^{(r-1)}| = |\lambda^{(r)}| + |\lambda^{(r-3)}| + n_{r-2} \times k$.

Hence $\lambda^{(r-1)}+((-k)^{n_{r-2}})=\xi$ and k should be nonpositive unless $n_{r-1}=n_{r-2}$.

Let ξ be a Young diagram of form $\xi=((i_{r-2})^{n_{r-2}})+((i_{r-1})^{n_{r-1}})$, where i_{r-2} and i_{r-1} are nonnegative integers. Let $\lambda^{(r-3)}$ be any Young diagram contained in ξ. We must show the following lemma.

LEMMA 5. *Using the same notations as above,*

$$LR^{\xi}_{\lambda^{(r-3)},\,((i_r))^{n_r})}=0 \text{ or } 1.$$

Proof. Let μ be a Young diagram defined by

$$\begin{aligned}
\mu_1 &= i_{r-1}+i_{r-2}-\lambda^{(r-3)}_{n_{r-1}},\\
\mu_2 &= i_{r-1}+i_{r-2}-\lambda^{(r-3)}_{n_{r-1}-1},\\
\vdots &= \qquad\vdots\\
\mu_{n_{r-1}} &= i_{r-1}+i_{r-2}-\lambda^{(r-3)}_{1},
\end{aligned}$$

and let ν be a Young diagram defined by

$$\begin{aligned}
\nu_1 &= i_{r-2}-\lambda^{(r-3)}_{n_{r-2}},\\
\nu_2 &= i_{r-2}-\lambda^{(r-3)}_{n_{r-2}-1},\\
\vdots &= \qquad\vdots\\
\nu_{n_{r-2}-n_{r-1}} &= i_{r-2}-\lambda^{(r-3)}_{n_{r-1}+1}.
\end{aligned}$$

If we recall that ξ is given by $\xi=((i_{r-2}+i_{r-1})^{n_{r-1}}, (i_{r-2})^{n_{r-2}-n_{r-1}})$, then the lengths of the rows of the skew diagram $\xi/\lambda^{(r-3)}$ are given from top to bottom by

$$\mu_{n_{r-1}}, \mu_{n_{r-1}-1}, \ldots, \mu_1, \nu_{n_{r-2}-n_{r-1}}, \nu_{n_{r-2}-n_{r-1}-1}, \ldots, \nu_1.$$

In the first n_{r-1} rows of the skew diagram $\xi/\lambda^{(r-3)}$, the symbols $1, 2, \ldots, n_r$ can be inserted uniquely to satisfy the Littlewood–Richardson rules; namely,

$$\begin{array}{cccccccccc}
 & & & & & & 1 & 1 & 1 \\
 & & & & 1 & 1 & 2 & 2 & 2 \\
 & & & & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 3 & 3 & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 2 & \vdots & \vdots & n_{r-1}-1 & n_{r-1}-1 & n_{r-1}-1 \\
\vdots & \vdots & \vdots & \vdots & n_{r-1}-1 & n_{r-1}-1 & n_{r-1} & n_{r-1} & n_{r-1}\ .
\end{array}$$

Therefore the symbol 1's occur μ_1 times and the symbol 2's occur μ_2 times and ... the symbol n_{r-1}'s occur $\mu_{n_{r-1}}$ times in this diagram.

Since each of the symbols $1, 2, \ldots, n_r$ must occur i_r times in the skew diagram $\xi/\lambda^{(r-3)}$, in the last row of $\xi/\lambda^{(r-3)}$ only the symbol n_r's must be inserted and the other symbols must not appear in that row.

The same argument shows that in the $(n_{r-2}-1)$-th row of the skew diagram $\xi/\lambda^{(r-3)}$, only the symbol $n_r - 1$'s occur and so on. Therefore the symbols $1, 2, \ldots, n_r$ can be inserted uniquely, satisfying the Littlewood–Richardson rules, in the last $n_{r-2}-n_{r-1}$ rows of the skew diagram $\xi/\lambda^{(r-3)}$; namely,

$$
\begin{array}{cccccccc}
 & & & & & n_r-n_{r-2}+n_{r-1}-1 & n_r-n_{r-2}+n_{r-1}-1 & n_r-n_{r-2}+n_{r-1}-1 \\
 & & & & & n_r-n_{r-2}+n_{r-1}-2 & n_r-n_{r-2}+n_{r-1}-2 & n_r-n_{r-2}+n_{r-1}-2 \\
 & & & & & \vdots & \vdots & \vdots \\
 & & & & n_r-2 & n_r-2 & n_r-2 & n_r-2 \\
 & & n_r-1 & n_r-1 & n_r-1 & n_r-1 & n_r-1 & n_r-1 \\
n_r & n_r & n_r & n_r & n_r & n_r & n_r & n_r
\end{array}
$$

Therefore the symbol n_r's occur ν_1 times and the symbol $n_r - 1$'s occur ν_2 times, ... and the symbol $n_r - n_{r-2} + n_{r-1} + 1$'s occur $\nu_{n_{r-2}-n_{r-1}}$ times in this diagram.

From the above argument, $\mu^{\dagger(i_r, n_r)}$ must be equal to ν and we have at most one way to insert the symbols in the skew diagram $\xi/\lambda^{(r-3)}$ satisfying the Littlewood–Richardson rules. The lemma is proved.

In this case we say two columns in $\lambda^{(r-3)}$ are complementary to each other if the sum of the lengths of these two columns is equal to $n_{r-2} + n_{r-1} - n_r$. We also say a column in $\lambda^{(r-3)}$ is complementary to a column attached at the vertex r if the sum of these two columns is equal to n_{r-2}.

The admissible sequence of Young diagrams is given as follows: The sequences of Young diagrams $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ corresponding to the relative invariants $S(V)^{\mathrm{rel}}$ in $S(V)$ are given as follows.

(RS1) $\lambda^{(1)} = (i_1^{n_1})$, and inductively $\lambda^{(s)}$ $(s \geqq 2)$ are obtained as follows.

If (1) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \to \overset{s+1}{\bullet}$ or (2) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{(s-1)} + (i_s^{n_s})$.

If (3) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$ or (4) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \to \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{\dagger(i_s, n_s)}$, where $i_s \in \mathbb{Z}$.

$$\lambda^{(r-2)} = \xi = ((i_{r-2})^{n_{r-2}}) + ((i_{r-1})^{n_{r-1}}),\ \lambda^{(r-1)} = (i_{r-1}^{n_{r-1}}),\ \lambda^{(r)} = (i_r^{n_r}).$$

(RS2) $\lambda^{(s)}$ is a Young diagram for any s.

(RS3) $l(\lambda^{(k)}) \leqq \text{minimum}(n_k, n_{k+1})$ for any $k = 1, 2, \ldots, r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$.

(RS4) $LR^{\xi}_{\lambda^{(r-3)}, \lambda^{(r)}} \neq 0$.

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ satisfying all the above conditions "admissible sequences of Young diagrams (for the relative invariants)."

For each admissible sequence, we have only one (up to scalars) relative invariant in $S(V)$. Also, a relative invariant is determined by its weight up to scalars.

We trace a column in an admissible sequence. Let this column attach at the vertex p and assume that this column survives until the vertex $r-2$. Then there exists the complementary column from Lemma 5 which attached at the vertex q.

For example, let the sinks and sources between p and $r-2$ be located in the following way:

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < v_s < u_s < r-2.$$

Then we have three cases.

Case 1. $n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q + n_r > n_{r-2}$ and $n_{r-1} > n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p$.

Then we define the matrix M by

$$M = \begin{pmatrix}
M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,u_s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{r-2,u_s} & M_{r-2,r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{r-1,r-2}M_{r-2,r} & M_{r-1,r-2}M_{r-2,u_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_t,u_t} & M_{v_t,q}
\end{pmatrix}.$$

Since $n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q+n_r+n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p=n_{r-2}+n_{r-1}$, M is a square matrix. This $\phi_{q,p,r-1,r}=\det(M)$ is called primitive if

$$\begin{aligned}
n_p &< n_{p+1}, n_{p+2}, \ldots, n_{v_1},\\
n_{v_1}-n_p &< n_{v_1+1}, n_{v_1+2}, \ldots, n_{u_1},\\
n_{u_1}-n_{v_1}+n_p &< n_{u_1+1}, n_{u_1+2}, \ldots, n_{v_2},\\
\vdots \quad &< \quad \vdots\\
n_{u_s}-n_{v_s}+\cdots+n_p &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}
\end{aligned}$$

and

$$\begin{aligned}
n_q &< n_{q+1}, n_{q+2}, \ldots, n_{v_t},\\
n_{v_t}-n_q &< n_{v_t+1}, n_{v_t+2}, \ldots, n_{u_t},\\
\vdots \quad &< \quad \vdots\\
n_{u_s}-n_{v_s}+\cdots+n_q &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}.
\end{aligned}$$

Case 2. $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p+n_r=n_{r-2}$.
Then $n_{u_s}-n_{v_s}+\cdots+n_{u_t}-n_{v_t}+n_q=n_{r-1}$ and this case reduces to A_r-type quivers.

Case 3. $n_{u_s}-n_{v_s}+\cdots+n_{u_1}-n_{v_1}+n_p+n_r=n_{r-2}+n_{r-1}$.
Then $n_r>n_{r-1}$ and M is given by

$$M=\begin{pmatrix}
M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0\\
0 & \ddots & \ddots & 0 & 0\\
0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,u_s} & 0\\
0 & 0 & 0 & M_{r-2,u_s} & M_{r-2,r}\\
0 & 0 & 0 & 0 & M_{r-1,r-2}M_{r-2,r}
\end{pmatrix}.$$

This $\phi_{p,r-1,r}=\det(M)$ is called primitive if

$$\begin{aligned}
n_p &< n_{p+1}, n_{p+2}, \ldots, n_{v_1},\\
n_{v_1}-n_p &< n_{v_1+1}, n_{v_1+2}, \ldots, n_{u_1},\\
n_{u_1}-n_{v_1}+n_p &< n_{u_1+1}, n_{u_1+2}, \ldots, n_{v_2},\\
\vdots \quad &< \quad \vdots\\
n_{u_s}-n_{v_s}+\cdots+n_p &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}.
\end{aligned}$$

We also define the primitive invariants for the other cases in which the sinks and sources between p and $r-2$ are located in different ways.

Then the theorem is as follows.

THEOREM. *The relative invariants in* $S(V)$ *amount to being the monomials in all the primitive determinantal invariants* $\phi_{q,p,r-1,r}$*'s*, $\phi_{p,r-1,r}$*'s*, $P_{q,p}$*'s, and the primitive relative invariants are algebraically independent.*

Proof. The proof is the same used before, so we omit it.

Next let F be a quiver defined by

Case: Ordinary at $r-2$ (only 2 arrows started from $r-2$ to $r-1$ and from $r-2$ to $r-3$).

$$\begin{tikzcd} \bullet \arrow[r] & \overset{p}{\bullet} \arrow[r] & \overset{v_1}{\bullet} & \overset{u_1}{\bullet} \arrow[l] & \bullet \arrow[l] \arrow[r] & \cdots & \bullet \arrow[l] & \overset{r-2}{\bullet} \arrow[l] \arrow[r] & \overset{r-1}{\bullet} \\ & & & & & & & \underset{r}{\bullet} \arrow[u] & \end{tikzcd}$$

Then we must determine when the linear character $(\det)^k$ of $GL(n_{r-2})$ occurs in the space $(V^{GL(n_{r-2})}_{\lambda^{(r-3)}})^* \otimes (V^{GL(n_{r-2})}_{\lambda^{(r-1)}})^* \otimes V^{GL(n_{r-2})}_{\lambda^{(r)}}$

Since

$$\begin{aligned} &\mathrm{Hom}_{GL(n_{r-2})}((\det)^k, (V_{\lambda^{(r-3)}})^* \otimes (V_{\lambda^{(r-1)}})^* \otimes V_{\lambda^{(r)}}) \\ &\qquad \cong \mathrm{Hom}_{GL(n_{r-2})}(V_{\lambda^{(r-3)}} \otimes V_{\lambda^{(r-1)}}, (\det)^{(-k)} \otimes V_{\lambda^{(r)}}) \end{aligned}$$

and $n_{r-2} \geqq n_r$, if $n_{r-2} > n_r$, $-k$ must be non-negative.

Lemma 5 tells us that k is determined uniquely and

$$\dim \mathrm{Hom}_{GL(n_{r-2})}(V_{\lambda^{(r-3)}} \otimes V_{\lambda^{(r-1)}}, (\det)^{(-k)} \otimes V_{\lambda^{(r)}}) \leqq 1.$$

Therefore the admissible sequences of Young diagrams corresponding to the relative invariants are given in a similar way as follows:

(RS1) $\lambda^{(1)} = (i_1^{n_1})$, and inductively $\lambda^{(s)}$ $(s \geqq 2)$ are obtained as follows.

If (1) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \to \overset{s+1}{\bullet}$ or (2) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{(s-1)} + (i_s^{n_s})$.

If (3) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$ or (4) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \to \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{\dagger(i_s, n_s)}$, where $i_s \in \mathbb{Z}$.

$\lambda^{(r-2)} = \xi = ((i_{r-2})^{n_{r-2}}) + ((i_r)^{n_r})$, $\lambda^{(r-1)} = (i_{r-1}^{n_{r-1}})$, $\lambda^{(r)} = (i_r^{n_r})$.

(RS2) $\lambda^{(s)}$ is a Young diagram for any s.

(RS3) $l(\lambda^{(k)}) \leqq \mathrm{minimum}(n_k, n_{k+1})$ for any $k = 1, 2, \ldots, r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$.

(RS4) $LR^{\xi}_{\lambda^{(r-3)}, \lambda^{(r-1)}} \neq 0$.

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$ satisfying all the above conditions "admissible sequences of Young diagrams (for the relative invariants)." For each admissible sequence, we have only one (up to scalars) relative invariants in $S(V)$.

We also trace a column in an admissible sequence. Let this column attach at the vertex p and assume that there exists a complementary column in Lemma 4 which attached at the vertex q.

Let the sinks and sources between p and $r-2$ be located in the following way:

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < u_{s-1} < v_s < r-2.$$

We have two cases.

Case 1. $n_{v_s} - n_{u_{s-1}} + \cdots - n_{u_t} + n_{v_t} - n_q + n_{r-1} > n_{r-2}$.

Then $n_r > n_{v_s} - n_{u_{s-1}} + \cdots - n_{u_1} + n_{v_1} - n_p$ since $n_{v_s} - n_{u_{s-1}} + \cdots - n_{u_t} + n_{v_t} - n_q + n_{r-1} + n_{v_s} - n_{u_{s-1}} + \cdots - n_{u_1} + n_{v_1} - n_p = n_{r-2} + n_r$.

We define the matrix M by

$$M = \begin{pmatrix} M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,r-2} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & M_{r-1,r-2} & M_{r-1,r-2}M_{r-2,r} & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & M_{v_s,r-2}M_{r-2,r} & M_{v_s,u_{s-1}} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_t,u_t} & M_{v_t,q} \end{pmatrix}.$$

This $\phi_{q,p,r-1,r} = \det(M)$ is called primitive if

$$\begin{aligned} n_p &< n_{p+1}, n_{p+2}, \ldots, n_{v_1}, \\ n_{v_1} - n_p &< n_{v_1+1}, n_{v_1+2}, \ldots, n_{u_1}, \\ n_{u_1} - n_{v_1} + n_p &< n_{u_1+1}, n_{u_1+2}, \ldots, n_{v_2}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s} - n_{v_s} + \cdots + n_p &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2} \end{aligned}$$

and

$$\begin{aligned} n_q &< n_{q+1}, n_{q+2}, \ldots, n_{v_t}, \\ n_{v_t} - n_q &< n_{v_t+1}, n_{v_t+2}, \ldots, n_{u_t}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s} - n_{v_s} + \cdots + n_q &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}. \end{aligned}$$

Case 2. $n_{v_s} - n_{u_{s-1}} + \cdots - n_{u_1} + n_{v_1} - n_p + n_{r-1} = n_{r-2}$. This case reduces the A_r-type quivers.

We also define the primitive invariants for the other cases in which the sinks and sources between p and $r-2$ are located in different ways. So the theorem is the same as before.

THEOREM. *The relative invariants in $S(V)$ amount to being the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $\phi_{p,r-1,r}$'s, $P_{q,p}$'s, and the primitive relative invariants are algebraically independent.*

Proof. The proof is just the same as before, so we omit it.

Case: Sink at $r-2$. Let F be a quiver of type D_n whose branching vertex is a sink, namely

$$\begin{tikzcd}[column sep=small]
\overset{p}{\bullet} \arrow[r] & \bullet \arrow[r] & \overset{v_1}{\bullet} & \cdots \arrow[l] \arrow[r] & \overset{q}{\bullet} \arrow[r] & \overset{v_t}{\bullet} & \cdots \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \overset{r-2}{\bullet} & \overset{r-1}{\bullet} \arrow[l] \\
 & & & & & & & & & \underset{r}{\bullet} \arrow[u] &
\end{tikzcd}$$

Then we must determine when the linear character $(\det)^k$ of $GL(n_{r-2})$ occurs in the space $V^{GL(n_{r-2})}_{\lambda^{(r-3)}} \otimes V^{GL(n_{r-2})}_{\lambda^{(r-1)}} \otimes V^{GL(n_{r-2})}_{\lambda^{(r)}}$.

If we put

$$V^{GL(n_{r-2})}_{\lambda^{(r-1)}} \otimes V^{GL(n_{r-2})}_{\lambda^{(r)}} = \sum_{\xi} LR^{\xi}_{\lambda^{(r-1)},\lambda^{(r)}} V^{GL(n_{r-2})}_{\xi}$$

and $\lambda^{(r-2)} = \xi$, then from Lemma 3, the Littlewood–Richardson coefficient $LR^{\xi}_{(i_{r-1}^{n_{r-1}})(i_r^{n_r})}$ is 0 or 1 and if $(\det)^k$ of $GL(n_{r-2})$ occurs in the above space, then $(\lambda^{(r-3)})^{\dagger(i_{r-2}, n_{r-2})} = \lambda^{(r-2)}$.

The admissible sequences of Young diagrams corresponding to the relative invariants are given as follows.

(RS1) $\lambda^{(1)} = (i_1^{n_1})$, and inductively $\lambda^{(s)}$ $(s \geqq 2)$ are obtained as follows.

If (1) $\overset{s-1}{\bullet} \rightarrow \overset{s}{\bullet} \rightarrow \overset{s+1}{\bullet}$ or (2) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{(s-1)} + (i_s^{n_s})$.

If (3) $\overset{s-1}{\bullet} \rightarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$ or (4) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \rightarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{\dagger(i_s, n_s)}$, where $i_s \in \mathbb{Z}$.

$\lambda^{(r-2)} = \xi = (\lambda^{(r-3)})^{\dagger(i_{r-2}, n_{r-2})}$, $\lambda^{(r-1)} = (i_{r-1}^{n_{r-1}})$, $\lambda^{(r)} = (i_r^{n_r})$, where $i_k \in \mathbb{Z}$.

(RS2) $\lambda^{(k)}$ is a Young diagram for any $k = 1, 2, ..., r$.

(RS3) $l(\lambda^{(k)}) \leqq \text{minimum}(n_k, n_{k+1})$ for any $k = 1, 2, ..., r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$.

(RS4) $LR^{\xi}_{\lambda^{(r-1)},\lambda^{(r)}} \neq 0$.

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(r)}$ satisfing all the above conditions "admissible sequences of Young diagrams (for the relative invariants)." For each admissible sequence, we have only one (up to scalars) relative invariant in $S(V)$.

We trace a column in an admissible sequence. Let this column attach at the vertex p and assume that there is a complementary column in Lemma 5 which is attached at the vertex q.

For example, let the sinks and sources between p and $r-2$ be located as.

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < v_s < u_s < r-2,$$

and satisfy the equality $n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q + n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p + n_{r-1} + n_r = 2n_{r-2}$.

Then we define the matrix M by

$$M = \begin{pmatrix}
M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,u_s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{r-2,u_s} & M_{r-2,r-1} & M_{r-2,r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{r-2,r} & M_{r-2,u_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_t,u_t} & M_{v_t,q}
\end{pmatrix}.$$

This $\phi_{q,p,r-1,r} = \det(M)$ is called primitive if

$$\begin{aligned}
n_p &< n_{p+1}, n_{p+2}, ..., n_{v_1}, \\
n_{v_1} - n_p &< n_{v_1+1}, n_{v_1+2}, ..., n_{u_1}, \\
n_{u_1} - n_{v_1} + n_p &< n_{u_1+1}, n_{u_1+2}, ..., n_{v_2}, \\
\vdots \quad &< \quad \vdots \\
n_{u_s} - n_{v_s} + \cdots + n_p &< n_{u_s+1}, n_{u_s+2}, ..., n_{r-2}
\end{aligned}$$

and

$$
\begin{aligned}
n_q &< n_{q+1}, n_{q+2}, \ldots, n_{v_t}, \\
n_{v_t} - n_q &< n_{v_t+1}, n_{v_t+2}, \ldots, n_{u_t}, \\
\vdots \quad &< \quad \vdots \\
n_{u_s} - n_{v_s} + \cdots + n_q &< n_{u_s+1}, n_{u_s+2}, \ldots, n_{r-2}.
\end{aligned}
$$

We also define the primitive invariants for the other cases in which the sinks and sources between p and $r-2$ are located in different ways. The theorem is the same as before.

THEOREM. *The relative invariants in $S(V)$ amount to being the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s, and the primitive relative invariants are algebraically independent.*

Finally, let F be a type-D_n quiver whose branching vertex is a source, namely

Case: Source at $r-2$.

$$
\begin{tikzcd}
\overset{p}{\bullet} \arrow[r] & \bullet \arrow[r] & \overset{v_1}{\bullet} & \cdots \arrow[l] \arrow[r] & \overset{q}{\bullet} \arrow[r] & \overset{v_t}{\bullet} & \cdots \arrow[l] \arrow[r] & \bullet \arrow[r] & \overset{v_s}{\bullet} & \overset{r-2}{\bullet} \arrow[l] \arrow[r] \arrow[d] & \overset{r-1}{\bullet} \\
 & & & & & & & & & r &
\end{tikzcd}
$$

Then we must determine when the linear character $(\det)^k$ of $GL(n_{r-2})$ occurs in the space $(V^{GL(n_{r-2})}_{\lambda^{(r-3)}})^* \otimes (V^{GL(n_{r-2})}_{\lambda^{(r-1)}})^* \otimes (V^{GL(n_{r-2})}_{\lambda^{(r)}})^*$.

If we put

$$V^{GL(n_{r-2})}_{\lambda^{(r-3)}} \otimes V^{GL(n_{r-2})}_{\lambda^{(r-1)}} = \sum_{\xi} LR^{\xi}_{\lambda^{(r-1)}, \lambda^{(r)}} V^{GL(n_{r-2})}_{\xi}$$

and $\lambda^{(r-2)} = \xi$, then from Lemma 1 the Littlewood–Richardson coefficient $LR^{\xi}_{(i^{n_{r-1}}_{r-1})(i^{n_r}_r)}$ is 0 or 1. If $(\det)^k$ of $GL(n_{r-2})$ occurs in the above space, then $(\lambda^{(r-3)})^{\dagger(i_{r-2}, n_{r-2})} = \lambda^{(r-2)}$.

The admissible sequences of Young diagrams corresponding to the relative invariants are given as follows.

(RS1) $\lambda^{(1)} = (i^{n_1}_1)$, and inductively $\lambda^{(s)}$ $(s \geqq 2)$ are obtained as follows.

If (1) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \to \overset{s+1}{\bullet}$ or (2) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{(s-1)} + (i^{n_s}_s)$.

If (3) $\overset{s-1}{\bullet} \to \overset{s}{\bullet} \leftarrow \overset{s+1}{\bullet}$ or (4) $\overset{s-1}{\bullet} \leftarrow \overset{s}{\bullet} \to \overset{s+1}{\bullet}$, then $\lambda^{(s)} = \lambda^{\dagger(i_s, n_s)}$, where $i_s \in \mathbb{Z}$.

$\lambda^{(r-2)} = \xi = (\lambda^{(r-3)})^{\dagger(i_{r-2}, n_{r-2})}$, $\lambda^{(r-1)} = (i^{n_{r-1}}_{r-1})$, $\lambda^{(r)} = (i^{n_r}_r)$, where $i_k \in \mathbb{Z}$.

(RS2) $\lambda^{(k)}$ is a Young diagram for any $k = 1, 2, \ldots, r$.

(RS3) $l(\lambda^{(k)}) \leqq \text{Minimum}(n_k, n_{k+1})$ for any $k = 1, 2, ..., r-3$ and $l(\lambda^{(r-2)}) \leqq n_{r-2}$.

(RS4) $LR^{\xi}_{\lambda^{(r-1)}, \lambda^{(r)}} \neq 0$.

We call the sequences $\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(r)}$ satisfying all the above conditions "admissible sequences of Young diagrams (for the relative invariants)." For each admissible sequences, we have only one (up to scalars) relative invariant in $S(V)$.

We trace a column in an admissible sequence. Let this column attach at the vertex p and assume that there is a complementary column in Lemma 4 which is attached at the vertex q.

For example, let the sinks and sources between p and $r-2$ be located in the following way:

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < u_{s-1} < v_s < r-2,$$

and satisfy $n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q + n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p + n_{r-1} + n_r = 2n_{r-2}$.

Then we define the matrix M by

$$M = \begin{pmatrix} M_{v_1, p} & M_{v_1, u_1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{v_s, u_{s-1}} & M_{v_s, r-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{r-1, r-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{r, r-2} & M_{r, r-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{v_s, r-2} & M_{v_s, u_{s-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_t, u_t} & M_{v_t, q} \end{pmatrix}.$$

This $\phi_{q, p, r-1} = \det(M)$ is called primitive if

$$\begin{aligned} n_p &< n_{p+1}, n_{p+2}, ..., n_{v_1}, \\ n_{v_1} - n_p &< n_{v_1+1}, n_{v_1+2}, ..., n_{u_1}, \\ n_{u_1} - n_{v_1} + n_p &< n_{u_1+1}, n_{u_1+2}, ..., n_{v_2}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s} - n_{v_s} + \cdots + n_p &< n_{u_s+1}, n_{u_s+2}, ..., n_{r-2} \end{aligned}$$

and

$$\begin{aligned} n_q &< n_{q+1}, n_{q+2}, ..., n_{v_t}, \\ n_{v_t} - n_q &< n_{v_t+1}, n_{v_t+2}, ..., n_{u_t}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s} - n_{v_s} + \cdots + n_q &< n_{u_s+1}, n_{u_s+2}, ..., n_{r-2}. \end{aligned}$$

We also define the primitive invariants for the other cases in which the sinks and sources between p and $r-2$ are located in different ways. Then the theorem is the same as before.

THEOREM. *The relative invariants in $S(V)$ amount to being the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s, and the primitive relative invariants are algebraically independent.*

Proof. The proof is just the same as that of the other cases, so we omit it.

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